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Characterizing Local Best SAIN Approximations

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1. INTRODUCTION

Let $f \in C^{(1)}[a, b]$ and let $Z = \{z_1, z_2, ..., z_k\}$, $a \leqslant z_1 < z_2 < \cdots < z_k \leqslant b$ be given. Let Π_{n-1} denote the set of algebraic polynomials of degree less than or equal to $n-1$. Let $\Pi_{n-1}(Z, f) \equiv \{p \in \Pi_{n-1}: p(x) = f(x) \quad \forall x \in Z, \|p\| = \|f\|\}$, where $\|\cdot\|$ denotes the uniform norm on $[a, b]$. This set is unfortunately not in general convex. We seek to characterize those $p^* \in \Pi_{n-1}(Z, f)$ for which $\|p^* - f\| \leqslant \|p - f\|$ for all $p \in \Pi_{n-1}(Z, f) \cap N$, where N is some neighborhood of p^*. Such a p^* will be called a local best SAIN approximation. (SAIN is an acronym for "Simultaneous Approximation, Interpolation and Norm preservation.") If $\Pi_{n-1}(Z, f) \neq \varnothing$, then we can use standard compactness arguments to show the existence of at least one globally best SAIN approximation.

To avoid trivial cases we will assume without additional comment that $f \notin \Pi_{n-1}(Z, f)$, and for certain technical reasons, we will also assume that the set of zeros of f' is composed of a finite number of (connected) components. The topology for functions is that induced by the uniform norm and the topology for point sets is the usual relative topology on $[a, b]$.

SAIN approximation was introduced by Deutsch and Morris [2] and has been subsequently studied in [5, 6, 10]. These investigations have been concerned with Jackson and Weierstrass type results and, as is pointed out by Chalmers and Taylor in their recent survey [1], no characterization results for this theory are known. In order to obtain such results, we find it useful to insist upon the conditions on f given above. Even under these restrictions the characterization theorem is exceedingly complex. The reader may wish to compare the theory that follows with that of Ross and Belford [11] concerning approximation with prescribed norm. The method of attack we use is similar to that in [9] but the details are quite different.

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2. Definitions and Notation

For $p \in \Pi_{n-1}(Z,f)$, let $e_p(x) \equiv p(x) - f(x)$, called the error function for the approximation. In this notation and in that which follows, we will include p as a subscript or argument for emphasis but remove it when the context is clear. Let $X_u(p) = \{x \in [a,b] : e_p(x) = \|e_p\|\}$, $X_l(p) = \{x \in [a,b] : e_p(x) = -\|e_p\|\}$ and $X(p) = X_u(p) \cup X_l(p)$. Let $Y_u(p) = \{x \in [a,b] : p(x) = \|f\|\}$, $Y_l(p) = \{x \in [a,b] : p(x) = -\|f\|\}$ and $Y(p) = Y_u(p) \cup Y_l(p)$. For sets of real numbers A and B, $A < B$ will mean that $a \in A$, $b \in B$ implies $a < b$, with a similar definition for $A \leqslant B$. Assume first that p is non-constant. Partition $(X \cup Y) - Z$ into a (finite) number of disjoint sets $\{W_i(p)\}_{i=1}^{m(p)}$, so that

(i) given W_i for any i, $z \in Z \Rightarrow W_i < \{z\}$ or $W_i > \{z\}$,

(ii) given $x \in X_u \cup Y_u$ and $y \in X_l \cup Y_l$, we must have $\{x, y\} \not\subset W_i$ for each $i = 1, 2, \ldots, m$,

(iii) the number of subsets in the partition is minimal (and clearly finite). This number is m.

This partitioning can be done in but one way. Note each W_i is closed. For $i = 1, 2, \ldots, m$, define $\sigma(W_i) = 1$ if $W_i \cap (X_u \cup Y_u) \neq \varnothing$ and define $\sigma(W_i) = -1$ otherwise. Let $I_0, I_1, \ldots, I_m$ be closed, non-singleton intervals, disjoint from $\bigcup_i W_i$, where $W_i < I_i < W_{i+1}$, for $i = 1, 2, \ldots, m-1$, and where $I_0 < W_1$ if $a \notin W_1$ and $I_m > W_m$ if $b \notin W_m$. I_0 or I_m will be empty if $a \in W_1$ or $b \in W_m$, respectively. The other I's are to be nonempty. We also require that $Z \subset \bigcup_{i=0}^{m} I_i$. Clearly, a collection of such I's can be found for each p. Define G_i for $i = 1, 2, \ldots, m-1$ by

$$
\begin{aligned}
G_i &= \operatorname{card}(Z-Y)\cap I_i && \text{if } \operatorname{card}(Z-Y)\cap I_i \text{ even and } \sigma(W_i)\sigma(W_{i+1}) = 1 \\
&= \operatorname{card}(Z-Y)\cap I_i + 1 && \text{if } \operatorname{card}(Z-Y)\cap I_i \text{ odd and } \sigma(W_i)\sigma(W_{i+1}) = 1 \\
&= \operatorname{card}(Z-Y)\cap I_i && \text{if } \operatorname{card}(Z-Y)\cap I_i \text{ odd and } \sigma(W_i)\sigma(W_{i+1}) = -1 \\
&= \operatorname{card}(Z-Y)\cap I_i + 1 && \text{if } \operatorname{card}(Z-Y)\cap I_i \text{ even and } \sigma(W_i)\sigma(W_{i+1}) = -1.
\end{aligned}
$$

Define $G_0 = \operatorname{card}(Z - Y) \cap I_0$ and $G_m = \operatorname{card}(Z - Y) \cap I_m$. Exceptionally, if p is identically constant define $G_i = 0$ for all i. Define $\gamma = 2\operatorname{card}(Z \cap Y) - \operatorname{card}(Z \cap Y \cap \{a, b\})$.

Definition. Let $y \in Y$. We say that y is buried to the right in X if there is a neighbourhood N of y such that $f(x) - \|e\| \geqslant -\|f\|$ for all $x \in N - [a, y]$ when $p(y) = -\|f\|$ or $f(x) + \|e\| \leqslant \|f\|$ for all $x \in N - [a, y]$ when $p(y) = \|f\|$. (It is possible for $N - [a, y]$ to be empty.) Similarly, we say that y is buried to the left in X if there is a neighborhood N of y such that

$f(x)-\|e\|\geqslant-\|f\|$ for all $x\in N-[y,b]$ when $p(y)=-\|f\|$ or $f(x)+\|e\|\leqslant \|f\|$ for all $x\in N-[y,b]$ when $p(y)=\|f\|$. If y is not buried (or unburied) to the right in X then $y\neq b$ and there is a neighborhood N of y such that $f(x)-\|e\|<-\|f\|$ or $f(x)+\|e\|>\|f\|$ for all $x\in N-[a,y]$. This is a consequence of the hypothesis on the zeros of f'. A similar statement can be made regarding unburied to the left. If y is buried on both sides in X, we say y is buried in X. Otherwise, we say y is unburied in X. Note that if y is buried in X then $y\in X$. We say Y is buried in X if y is buried in X for each $y\in Y$.

Define $\beta(y)$ for each $y\in Y$ not buried in X as follows. If for some i, $1\leqslant i\leqslant m-1$, we have $X\cap W_i\leqslant\{y\}\leqslant X\cap W_{i+1}$ and y is not buried in X on the side of y adjacent to I_i, then $B(y)$ is given by Table I. If for some i, $1\leqslant i\leqslant m-1$, we have $X\cap W_i\leqslant\{y\}\leqslant X\cap W_{i+1}$ and y is buried in X on the side of y adjacent to I_i, define $\beta(y)=2$. If there exist x_1, x_2 in some $X\cap W_i$ (where $1\leqslant i\leqslant m$) with $x_1<y<x_2$, define $\beta(y)=2$. If $\{y\}\leqslant X\cap W_0$ and y is unburied to the left in X or $\{y\}\geqslant X\cap W_m$ and y is unburied to the right in X, define $\beta(y)=1$. If $\{y\}\leqslant W_0$ and y is buried to the left in X or $\{y\}\geqslant W_m$ and y is buried to the right in X, define $\beta(y)=2$, unless $\{y\}\cap\{a\}-X\neq\varnothing$ or $\{y\}\cap\{b\}-X\neq\varnothing$, respectively, in which case define $\beta(y)=1$. Note: If $X\cap W_i\cap Y=\{y\}$ is a singleton, it is possible that $\beta(y)$ is ambiguously defined. In this instance we agree to define β using the smaller value.

Define α by

$$\begin{aligned}\alpha &= \min\{\beta(y): y\in Y, y \text{ unburied in } X\} \qquad \text{if} \quad Y\cap Z=\varnothing\\ &= 0 \qquad \text{if} \quad Y\cap Z\neq\varnothing \text{ or p identically constant.}\end{aligned}$$

After this extensive series of definitions, we are ready to present the main theorem.

TABLE I

$\sigma(W_i)\,\sigma(W_{i+1})$	card$(Z-Y)\cap I_i$ Even	Odd
1	$\beta=2$	$\beta=0$
-1	$\beta=0$	$\beta=2$

3. CHARACTERIZATION

THEOREM 1. *Assume $p^* \in \Pi_{n-1}(Z,f)$ and $Y(p^*)$ is not buried in $X(p^*)$. Then p^* is a local best SAIN approximation to f if and only if*

$$\sum_{i=0}^{m(p^*)} G_i(p^*) + \alpha(p^*) + \gamma(p^*) \geqslant n.$$

Proof (*Only if*). Assume here and in the proof of the converse that p^* is non-constant. The result follows in a straightforward manner using standard techniques otherwise. Assume that $\sum_{i=0}^{m} G_i + \alpha + \gamma < n$. Given $\varepsilon > 0$, we show how to construct a SAIN approximation $\bar{p}$ to f such that $0 < \|\bar{p} - p^*\| < \varepsilon$ and $\|\bar{p} - f\| < \|p^* - f\|$. We must consider a large number of cases and subcases. Case 1: $Y \cap Z = \varnothing$ (implying $\gamma = 0$) and $\alpha = 0$. From the definitions of α and β, we see that there are several ways that this situation can occur. Take, for example, the subcase where there is a $\bar{y} \in Y$, not buried to the right in X, and i^*, $1 \leqslant i^* \leqslant m-1$, such that $(X_u \cup Y_u) \cap W_{i^*} \leqslant \{\bar{y}\} < I_{i^*} < (X_l \cup Y_l) \cap W_{i^*+1}$, the sets involved being nonempty, with $\text{card}(I_i \cap Z)$ even. We may assume that $\bar{y}$ is maximal among all points in Y satisfying these conditions (for fixed i^*). For each $\delta \in [0, d(Z \cap I_{i^*}, \bar{y}))$, where d is the norm induced distance function between points and sets, define $q_\delta \in \Pi_{n-1}$ so as to satisfy the following:

(1) q_δ has nodal zeros at each point in Z. (See [7] for definitions of nodal and nonnodal zeros.)

(2) q_δ has a nodal zero at an arbitrary point in $I_i - Z$ provided that $1 \leqslant i \leqslant m-1$, $i \neq i^*$ and either $\text{card}(I_i \cap Z)$ odd and $\sigma(W_i)\,\sigma(W_{i+1}) = 1$ or else $\text{card}(I_i \cap Z)$ even and $\sigma(W_i)\,\sigma(W_{i+1}) = -1$. The arbitrary points are chosen independently of δ. (One might alternatively require exactly one zero of q_δ in $I_i \cap Z$ be nonnodal if $I_i \cap Z \neq \varnothing$, rather than introduce the extra point.)

(3) q_δ has a nodal zero at $\bar{y} + \delta$.

(4) $\text{sgn}(q_\delta(\hat{x})) = -\text{sgn}(e(\hat{x}))$ for some $\hat{x} \in X$ chosen arbitrarily, and independent of δ.

(5) q_δ has no zeros other than the ones specified above.

(6) Additional conditions are specified on q_δ, given independently of δ, so that q_δ varies continuously with δ and so that $q'_\delta(\bar{y} + \delta) \neq 0$ for all allowable δ. (This could be done by using a normalized polynomial of minimal degree or by specifying higher derivatives at one of the nodal zeros given above, such specifications being consistent with condition 4.)

We note that such a q_δ can be formed since the number of zero requirements counting nodal zeros once and nonnodal zeros twice, is exactly

$\sum_{i=0}^{m} G_i + \alpha + \gamma < n$. We form the approximation $p^* + \mu q_\delta$ for $\mu > 0$. It is clear that this approximation interpolates f on Z. We wish to show that μ and $\delta > 0$ can be chosen so that

(i) $\|p^* + \mu q_\delta\| = \|f\|$,

(ii) $\|\mu q_\delta\| < \varepsilon$,

(iii) $\|p^* + \mu q_\delta - f\| < \|p^* - f\|$.

Note first that $\|p^* + \mu q_0\| > \|f\|$ for any $\mu > 0$, since $q_0'(\bar{y}) \neq 0$. Indeed the zero structure of q_0 has been defined in such a way that $|(p^* + \mu q_0)(x)| > \|f\|$ for x in some deleted right neighborhood of $\bar{y}$ and $|(p^* + \mu q_0)(x)| < \|f\|$ for x in some deleted left neighborhood of $\bar{y}$. For $\delta > 0$, we may deduce from continuity considerations that there is a function ω: $(0, \infty) \to (0, \infty)$ such that $\|p^* + \mu q_\delta\| = \|f\|$ for $\mu = \omega(\delta)$, where we remark that $\omega(\delta) \to 0$ as $\delta \to 0$. But $\|q_\delta\| \to \|q_0\|$ as $\delta \to 0$ so $\|\omega(\delta)\, q_\delta\| \to 0$ as $\delta \to 0$. Using standard arguments, one can show that the zero structure of $\omega(\delta)\, q_\delta$ is such that for any neighborhood N of $\bar{y}$, $|(p^* + \omega(\delta)\, q_\delta - f)(x)| < \|p^* - f\|$ for $x \notin N \cap (\bar{y}, b]$, provided $\|\omega(\delta)\, q_\delta\|$ is sufficiently small. But the above equality is actually true for $x \in N \cap (\bar{y}, b]$ as well, if N is chosen appropriately, since $\bar{y}$ is unburied to the right in X. Indeed, let N be the neighbourhood whose existence is assured in the definition of $\bar{y}$ unburied to the right in X. Then for $x \in N \cap (\bar{y}, b]$, denoting $p^* + \omega(\delta)\, q_\delta$ by $\bar{p}$, $\bar{p}(x) \leqslant \|f\| < f(x) + \|e_{p^*}\|$ implying $(\bar{p} - f)(x) < \|e_{p^*}\|$. Since $\bar{p}(x) - f(x) > -\|e_{p^*}\|$ for all $x \in [a, b]$ and $\|\omega(\delta)\, q_\delta\|$ sufficiently small, we deduce that $|\bar{p}(x) - f(x)| < \|e_{p^*}\| = \|p^* = f\|$ for all $x \in N \cap (\bar{y}, b]$ and $\|\omega(\delta) q_\delta\|$ sufficiently small, and hence $|\bar{p}(x) - f(x)| < \|p^* - f\|$ for all $x \in [a, b]$ and $\|\omega(\delta) q_\delta\|$ sufficiently small, as required. Thus conditions (i)–(iii) may be satisfied for $\mu = \omega(\delta)$ and $\delta > 0$ sufficiently small. Therefore, p^* is not a local best SAIN approximation.

If we consider the two possible choices for $\sigma(W_{i^*})$, the two possible choices for $\sigma(W_{i^*})\sigma(W_{i^*+1})$ (i.e., $\sigma(W_{i^*})\, \sigma(W_{i^*+1}) = 1$ and $\text{card}(I_{i^*} \cap Z)$ odd or else $\sigma(W_{i^*})\, \sigma(W_{i^*+1}) = -1$ and $\text{card}(I_{i^*} \cap Z)$ even), and the two possible relative locations for $\bar{y}$ (i.e., $(X \cup Y) \cap W_{i^*} \leqslant \{\bar{y}\} < I_{i^*}$ and $I_{i^*} < \{\bar{y}\} \leqslant (X \cup Y) \cap W_{i^*}$), it is clear that we have considered but one of a total of 8 distinct subcases of Case 1. In fact, the construction just described handles the 4 cases where $\{\bar{y}\} < I_{i^*}$. By modifying condition (3) in this construction to (3′): q_δ has a nodal zero at $\bar{y} - \delta$, the remaining cases are handled. We assume $\bar{y}$ minimal rather than maximal in this situation. The proof for each subcase is similar to that of the sample subcase considered above. In each subcase, we note that $\bar{y}$ is unburied in X on the side of $\bar{y}$ adjacent to I_{i^*}.

Case 2. $Y \cap Z = \varnothing$, $\alpha = 2$. This implies $\beta(y) = 2$ for all $y \in Y$ not buried in X. We consider 4 subcases.

Subcase 2(a). $X \cap W_{i^*} \leqslant \{\bar{y}\} \leqslant X \cap W_{i^*+1}$, $1 \leqslant i^* \leqslant m-1$, $\bar{y}$ not buried in X on the side of $\bar{y}$ adjacent to I_{i^*}, and either $\sigma(W_{i^*})\,\sigma(W_{i^*+1}) = 1$, $\operatorname{card}(Z \cap I_{i^*})$ even or else $\sigma(W_{i^*})\,\sigma(W_{i^*+1}) = -1$, $\operatorname{card}(Z \cap I_{i^*})$ odd. The construction here is similar to that of Case 1 except that an additional nodal zero must be specified near $\bar{y}$ (for example, in $I_{i^*} - Z$). This zero does not depend on δ and is required to preserve the proper zero structure of the *correction* function. The proof then proceeds as in Case 1. We assume that $\bar{y}$ is maximal or minimal, as in Case 1, depending on its location relative to I_{i^*}.

Subcase 2(b). $X \cap W_{i^*} \leqslant \{\bar{y}\} \leqslant X \cap W_{i^*+1}$, $1 \leqslant i^* \leqslant m-1$, $\bar{y}$ buried in X on the side of $\bar{y}$ adjacent to I_{i^*}. The construction here is again similar to that of Case 1. A nodal zero is specified at $\bar{y} - \delta$ if $\{\bar{y}\} < I_{i^*}$ and at $\bar{y} + \delta$ if $\{\bar{y}\} > I_{i^*}$. An additional nodal zero must also be specified as follows: Assume that $\{\bar{y}\} < I_{i^*}$. Then there is a deleted left neighborhood of $\bar{y}$ that fails to intersect X. If there is a deleted left neighborhood of $\bar{y}$ that also fails to intersect Y, specify the additional nodal zero arbitrarily in this neighborhood and consider only δ's for which $\bar{y} - \delta$ exceeds this zero. If, however, no such neighborhood can be found, then there must be a point in Y that satisfies the conditions of Subcase 2(c) given below, and for which the construction given in that subcase is applicable. A similar procedure is adopted if $\{\bar{y}\} > I_{i^*}$. Again, a proof similar to that of Case 1 may be used; the extra zero is specified to maintain the correct zero structure.

Subcase 2(c). There exist x_1, x_2 in $X \cap W_{i^*}$, $1 \leqslant i^* \leqslant m$ with $x_1 < \bar{y} < x_2$. Assume that $\bar{y}$ is buried to the right in X. A similar procedure is used if $\bar{y}$ is buried to the left. If there is a deleted left neighborhood of $\bar{y}$ that fails to intersect Y, then the construction of Subcase 2(b) may be employed. If there are points of Y in every such neighborhood, then there is a point $\bar{\bar{y}} \in Y$, to the left of $\bar{y}$, with $x_1 < \bar{\bar{y}} < x_2$, and contained in a neighborhood that fails to intersect X. In this situation we use a variation of the construction in Case 1, where requirement (3) is replaced by (3′): q_δ has a nonnodal zero at $\bar{y}$. (The dependence on δ is then lost, so that requirement (6) becomes irrelevant.) Of course we may consider initially a $\bar{y}$ with the properties of $\bar{\bar{y}}$ and the construction is as above. The proof is an easier version of those above.

Subcase 2(d). $\{\bar{y}\} \leqslant W_0$, $\bar{y}$ is buried to the left in X and $\{\bar{y}\} \cap \{a\} - X = \varnothing$ or else $\{\bar{y}\} \geqslant W_m$, $\bar{y}$ is buried to the right in X and $\{\bar{y}\} \cap \{b\} - X = \varnothing$. This case is treated in a manner similar to Subcase 2(b).

Case 3. $Y \cap Z = \varnothing$, $\alpha = 1$. This case requires examination of the situation near the endpoints and is handled using a straightforward adaptation of the constructions of Cases 1 and 2.

Case 4. $Y \cap Z \neq \varnothing$. The construction here is somewhat different. We must of necessity form $\bar{p} = p^* + \mu q$, $\mu > 0$ with q having nonnodal zeros at each point in $Y \cap Z - \{a, b\}$ and nodal-zeros at each point in $Y \cap Z \cap \{a, b\}$. The additional zeros are specified as follows:

(1) q has nodal zeros at each point in $Z - Y$.

(2) q has a nodal zero at an arbitrary point in $I_i - Z$ provided $1 \leqslant i \leqslant m-1$ and either $\operatorname{card}(I_i \cap (Z - Y))$ odd and $\sigma(W_i)\, \sigma(W_{i+1}) = 1$ or else $\operatorname{card}(I_i \cap (Z - Y))$ even and $\sigma(W_i)\, \sigma(W_{i+1}) = -1$.

(3) $\operatorname{sgn}(q(\hat{x})) = -\operatorname{sgn}(e(\hat{x}))$ for some $\hat{x} \in X$ chosen arbitrarily.

(4) q has no zeros other than the ones specified above.

With this definition for q, we see, using standard arguments, that $\| p^* - f \| > \| p^* + \mu q - f \|$ for sufficiently small $\mu > 0$. We note that $\| p^* + \mu q \| = \| f \|$ for small μ is an immediate consequence of the zero structure of q and the fact that $Z \cap Y \neq \varnothing$. Since $p^* + \mu q$ still interpolates f on Z, we see that p^* cannot be locally best.

We conclude the proof of the *only if* portion of the theorem by noting that the pair of zeros specified near $\bar{y}$ in Subcase 2(a) may be replaced by a nonnodal zero at $\bar{y}$, if $\bar{y}$ happens not be a point in X. Also note that $\sum_{i=0}^{m} G_i$ represents the number of zeros required to improve the approximation while maintaining the interpolation conditions, α represents the number of additional zeros required to insure $\| \bar{p} \| = \| f \|$ and γ represents the correction necessary when higher multiplicity zeros are required for points in $Y \cap Z$.

Proof (*If*). Suppose $\sum_{i=0}^{m} G_i + \alpha + \gamma \geqslant n$ but p^* is not locally best. Then there is a sequence of SAIN approximations to f, $\langle p_j \rangle$, $p_j \to p^*$ with $\| p_j - f \| < \| p^* - f \|$ for all j. Let $q_j = p_j - p^*$. For sufficiently large j, q_j has a zero structure similar to one of the "improvement functions" constructed in the proof of the "only if" part of the theorem. Indeed q_j must have zeros at each point in Z with multiplicities of at least two at each point in $Y \cap Z - \{a, b\}$. In fact since $\| p_j - f \| < \| p^* - f \|$, q_j has at least $\sum_{i=1}^{m} G_i + \gamma$ zeros (counting multiplicities) required to improve the approximation while maintaining the interpolation conditions. From continuity considerations, there is a $\bar{y} \in Y$, unburied in X and a sequence $\langle \bar{y}_j \rangle$ such that $\bar{y}_j \to \bar{y}$ and $| p_j(\bar{y}_j) | = | p^*(\bar{y}) | = \| f \|$ for all j. By considerations similar to those in the proof of the "only if" portion of the theorem we may deduce that for sufficiently large j, q_j must have at least α zeros in addition to those previously counted. Thus for a sufficiently large j, call it j^*, q_{j^*} has at least $\sum_{i=1}^{n} G_i + \alpha + \gamma \geqslant n$ zeros (counting multiplicities). But $q_{j^*} \in \Pi_{n-1}$ and so $q_{j^*} \equiv 0$, a contradiction, since $\| p_{j^*} - f \| < \| p^* - f \|$ implies $q_{j^*} \not\equiv 0$.

In the statement of Theorem 1, we assumed that Y was unburied in X. We now investigate the consequences of the removal of that condition.

THEOREM 2. *Assume $p^* \in \Pi_{n-1}(Z,f)$ and Y is buried in X. Then p^* is a local best SAIN approximation to f.*

Proof. Suppose p^* is not locally best. Then there is a sequence of SAIN approximations $\langle p_j \rangle$, $p_j \to p^*$, such that $\|p_j - f\| < \|p^* - f\|$ for all j. As in the proof of Theorem 1, there is a $\bar{y} \in Y$ and a sequence $\langle \bar{y}_j \rangle$ in $[a, b]$ for which $\bar{y}_j \to \bar{y}$ and $|p_j(\bar{y}_j)| = |p^*(\bar{y})| = \|f\|$. Since Y is buried in X, $\bar{y}$ is buried in X. So there is a neighborhood N of $\bar{y}$ in which $f(x) - \|e\| \geq -\|f\|$ for all $x \in N$ if $p^*(\bar{y}) = -\|f\|$ or else $f(x) + \|e\| \leq \|f\|$ for all $x \in N$ if $p^*(\bar{y}) = \|f\|$. Assume the latter. Let j^* be so large that $\bar{y}_j^* \in N$ and $p_{j^*}(\bar{y}_{j^*}) = |p_{j^*}(\bar{y}_{j^*})| = \|f\| \geq f(\bar{y}_{j^*}) + \|e\|$, so $|p_j(\bar{y}_{j^*}) - f(\bar{y}_{j^*})| = p_{j^*}(\bar{y}_{j^*}) - f(\bar{y}_{j^*}) \geq \|e\|$, a contradiction. The other case is handled similarly.

We may combine Theorems 1 and 2 to obtain a complete characterization of local best SAIN approximations.

THEOREM 3. *$p^* \in \Pi_{n-1}(Z,f)$ is a local best SAIN approximation to f if and only if one of the following conditions holds:*

(1) *Y is buried in X,*

(2) $\sum_{i=0}^{m} G_i + \alpha + \gamma \geq n$.

4. NONUNIQUENESS

It would be very surprising in view of the local nature of the characterization theorem, the nonconvexity of $\Pi_{n-1}(Z,f)$ and the nonuniqueness results of Ross and Belford [11], to discover that (globally) best SAIN approximatons are always unique. Indeed the following simple example shows that this cannot be expected.

EXAMPLE. Define f on $[0, 1]$ by $f(x) = 2(x - \frac{1}{2})^2 + \frac{1}{2}$. Let $Z = \{\frac{1}{2}\}$. It is easy to show that $\Pi_1(Z,f) = \{p_1, p_2\}$, where $p_1(x) = x$ and $p_2(x) = 1 - x$. But $\|p_1 - f\| = \|p_2 - f\| = 1$, so there are two distinct best SAIN approximations in this instance.

5. CONCLUDING REMARKS

It would be advantageous to remove the smoothness requirement on f as well as the requirement concerning the zeros of f'. In fact these requirements can probably be weakened or even completely removed at the expense of introducing substantial additional complications in the theory.

The theory might be widened to include Hermite–Birkhoff interpolatory conditions or more general constraints given by linear functionals and indeed such constraints have been considered in the literature.

The characterization theorem given (Theorem 1) is not a true alternation theorem. But it is "alternation-like" in the sense that given an approximation p^*, one can identify certain types of points in $[a, b]$ and determine whether p^* is a local best approximation to f solely on the basis of the pattern that these points form in $[a, b]$.

The results in Theorems 1, 2 and 3 can be extended to certain classes of Tchebycheff space. For instance, these theorems remain valid if our original approximating family is taken to be $\text{sp}\{e^{\lambda_1 x}, e^{\lambda_2 x},..., e^{\lambda_n x}\}$, where the λ_i's are distinct reals. More generally, the three theorems remain true if the approximating family originates with a space spanned by an extended complete Tchebycheff system of order 2, provided the nonconstant elements of such a space assume any given value at only a finite number of points. The reader may wish to consult [3, 4, 7, 8] for the appropriate definitions, theorems and related results.

References

1. B. L. Chalmers and G. D. Taylor, Uniform approximation with constraints, *Jber. Dt. Math Verein.* **81** (1979), 49–86.
2. F. Deutsch and P. D. Morris, On Simultaneous approximation and interpolation which preserves the norm, *J. Approx. Theory* **2** (1969), 355–373.
3. W. Haussmann, Differentiable Tchebycheff subspaces and Hermite interpolation, *Ann. Acad. Sci. Fenn. Ser. A. I. Math.* **4** (1978/1979), 75–83.
4. Y. Ikebe, Hermite–Birkhoff interpolation problems in Haar subspaces, *J. Approx. Theory* **8** (1973), 142–149.
5. D. J. Johnson, Jackson type theorems for approximation with side conditions, *J. Approx. Theory* **12** (1974), 213–229.
6. D. J. Johnson, SAIN approximation in $C[a, b]$, *J. Approx. Theory* **17** (1976), 14–34.
7. S. Karlin and W. J. Studden, "Tchebycheff Systems with Applications in Analysis and Statistics," Interscience, New York/London/Sydney, 1966.
8. L. L. Keener, Hermite–Birkhoff interpolation and approximation of a function and its derivatives, *J. Approx. Theory* **30** (1980), 129–138.
9. L. L. Keener, One-sided approximation with interpolation, preprint.
10. J. Lambert, Conditions for simultaneous approximation and interpolation with norm preservation in $C[a, b]$, *Pacific J. Math.* **66** (1976), 173–180.
11. M. D. Ross and G. G. Belford, Characterization of best Chebyshev approximations with prescribed norm, *J. Approx. Theory* **16** (1976), 315–328.